Compositional Foundations of Finite-Dimensional Classical Mechanics: Synthesis, Obstructions, and an Empirical Contract

PRINCIPIA PHYSICA Pilot — Synthesis

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Abstract

We synthesize three reconstructions of finite-dimensional classical mechanics and test the founding thesis that physics studies empirically realized compositional mathematical structures. A Hamiltonian theory object must include not only (M, ω, H), but selected preparations, observables, processes, decomposition and a realization map. This enrichment prevents mathematical isomorphism from being mistaken for physical equivalence. It also exposes four obstructions: canonical relations compose only partially; interaction is not an invariant of (M, ω, H); symplectic Hamiltonian systems are not closed under ordinary interconnection; and finite-precision observational equivalence is neither transitive nor a congruence. Exact data do not in general identify a unique Hamiltonian. Consequently the founding thesis is not itself a falsifiable physical hypothesis. Its useful residue is an explicit empirical contract whose compositional clauses can fail. We retain all assumptions, counterexamples, domain restrictions, and unsettled novelty claims rather than absorbing them into a slogan.

1 Question, scope, and epistemic discipline

The pilot question is whether finite-dimensional Hamiltonian mechanics can be represented as an empirically interpreted compositional theory object without conflating mathematical and physical equivalence. Field theory, general relativity, quantum theory and infinite-dimensional analysis are boundary tests only. The mathematical foundations are standard [1, 2]; the synthesis joins the three topic studies [9, 10, 11].

Assumption 1.1 (Smooth finite scope)**.** State spaces are finite-dimensional, second-countable smooth manifolds. Vector fields are locally Lipschitz; either their relevant flows are complete or a finite observation horizon is fixed. These conditions exclude genuine failures: Newtonian many-body dynamics can develop noncollision singularities [15], while non-Lipschitz forces need not determine unique trajectories [8].

Axiom 1.2 (Epistemic separation)**.** Definitions fix vocabulary; axioms and assumptions are inputs; mathematical claims require proof; empirical statements require an instrument and data; established physical results require citations; interpretations do not inherit theorem status; conjectures remain open. No mathematical equivalence is a physical equivalence until preparations, observables, units, domains and measurement maps have been shown to correspond.

Interpretation 1.3. Axiom 1.2 is methodological, not a theorem about nature. It is the device that keeps the phrase "empirically realized" from laundering assumptions into results.

2 The reconstructed theory object

Definition 2.1 (Empirically interpreted compositional system)**.** An object is a tuple

$$\mathbb{T} = (M, \omega, H, \mathcal{A}, S, \mathcal{O}, \mathcal{P}, \mathcal{D}, I),$$

where (M, ω) is symplectic, $H \in C^\infty(M)$, $\mathcal{A} \subseteq C^\infty(M)$ is a Poisson algebra, $S \subseteq M$ is a preparation class, $\mathcal{O} \subseteq \mathcal{A}$ is the measured subfamily, $\mathcal{P}$ is a chosen process class, $\mathcal{D}$ is decomposition and interconnection data, and I maps prepared trajectories to possible finite records under an instrument specification. Hamiltonian evolution obeys $\iota_{X_H}\omega = dH$, $\dot{f} = \{f, H\}$.

Lemma 2.2 (Naturality of Hamiltonian evolution)**.** *If $\varphi : (M, \omega) \to (N, \eta)$ is symplectic and $H = K \circ \varphi$, then $T\varphi\, X_H = X_K \circ \varphi$ and, wherever defined, $\varphi \circ \Phi_H^t = \Phi_K^t \circ \varphi$.*

Proof. For every v, $\eta(T\varphi X_H, T\varphi v) = \omega(X_H, v) = dH(v) = dK(T\varphi v)$. Nondegeneracy gives the vector-field identity; uniqueness of integral curves gives the flow identity. $\square$

Proposition 2.3 (The bare triple forgets empirical structure)**.** *The functor that forgets $(\mathcal{A}, S, \mathcal{O}, \mathcal{P}, \mathcal{D}, I)$ is not faithful to physical distinctions: one triple can support inequivalent measured quantities, apparatus couplings and subsystem decompositions.*

Proof. On $(\mathbb{R}^2, dq \wedge dp)$ the symplectic map $(q, p) \mapsto (2q, p/2)$ conjugates suitable Hamiltonians by Lemma 2.2. If each laboratory nevertheless reads its first coordinate, the pointer values differ by a factor two. Thus conjugacy of the bare triples does not fix the realization map. Likewise a linear canonical change on $\mathbb{R}^4$ can turn two uncoupled equal oscillators into coordinates in which the same Hamiltonian contains cross terms; the triple does not select which variables denote parts [9]. $\square$

Empirical statement 2.4 (Units and access)**.** Laboratories measure bounded-range, finite-resolution pointer records, not arbitrary elements of $C^\infty(M)$. Preparations occupy regions rather than mathematical points. Treating every smooth function and every point as physically accessible is therefore an idealizing assumption, not an established fact.

3 Composition: what works and what fails

Established physical result 3.1 (Monoidal product)**.** Symplectic manifolds admit the symmetric monoidal product $(M, \omega) \otimes (N, \eta) = (M \times N, \omega \oplus \eta)$ with one-point unit. For noninteracting Hamiltonians, $H \otimes K = H \circ \mathrm{pr}_M + K \circ \mathrm{pr}_N$ [1].

Theorem 3.2 (The symplectic product is not cartesian)**.** *If $\dim M = 2n > 0$ and $\dim N = 2m > 0$, neither the graph of $\mathrm{pr}_M : M \times N \to M$ nor the graph of $x \mapsto (x, x)$ is a canonical relation of the required type. Hence the monoidal product supplies neither deletion nor copying.*

Proof. The projection graph has dimension $2n + 2m$, whereas a Lagrangian in $(M \times N)^- \times M$ has dimension $2n + m$; its pulled-back form is $-\eta \neq 0$. The diagonal graph has dimension $2n$, whereas a Lagrangian in $M^- \times M \times M$ has dimension $3n$; its pulled-back form is $-\omega + \omega + \omega = \omega \neq 0$. □

Counterexample 3.3 (Canonical relations do not always compose)**.** Canonical relations compose by a fibre product. Without clean intersection and an appropriate projection, the fibre product may be singular or its image nonembedded. Explicit generating functions yield a line with a half-ray attached at the origin, which is not a submanifold [9]. Thus SympRel is only a partially defined "category" [13, 14].

New result claimed by this work 3.4 (Interaction needs empirically supplied decomposition)**.** Within this reconstruction, "the two subsystems interact" is not an invariant of (M, ω, H). It is a predicate of the enriched pair $((M, \omega, H), \mathcal{D})$.

Proof. For equal oscillators, normal-mode canonical coordinates diagonalize a spring-coupled Hamiltonian. Relative to particle coordinates it contains a cross term and is called interacting; relative to normal modes the same Hamiltonian is a sum and the flow is a product. Since a symplectic automorphism connects the descriptions, no invariant of the bare triple distinguishes them. The designation becomes determinate only when $\mathcal{D}$ is tied to preparation and readout. □

Established physical result 3.5 (Closure requires a larger class)**.** Autonomous symplectic Hamiltonian systems are not closed under common physical interconnections. Nonholonomic constraints, dampers, variable-rank constraints and singular reduction respectively produce almost-Poisson dynamics, volume contraction, rank-changing presymplectic geometry and stratified spaces [5, 7, 12]. Dirac and port-Hamiltonian structures provide a broader class closed under regular power-conserving interconnection [4, 3]; singularity hypotheses still matter.

Corollary 3.6. *No single operation in the present finite-dimensional symplectic category is simultaneously total, expressive of arbitrary interaction, and closed on its objects.*

Proof. The direct product is total but represents noninteraction unless extra H_{int} is supplied; canonical-relation composition is expressive but partial by Counterexample 3.3; physical interconnection leaves the class by Established result 3.5. □

4 Empirical realization and equivalence

Definition 4.1 (Realization)**.** An instrument is $\mathcal{R} = (T, \Delta t, S, \mathcal{O}, \sigma, \varepsilon_{\text{read}}, D)$: duration, sampling interval, preparations, measured observables, preparation resolution, readout precision and bounded range. Its realization map assigns to each nominal preparation the set of sampled records generated from its σ-neighbourhood. A model is realized on a record when some allowed prediction lies within the stated tolerance and inside D.

Definition 4.2 (Observational tolerance)**.** For models sharing preparation and observable labels, let $\beta_{\mathcal{O}} : f \mapsto f'$ and $\beta_S : s \mapsto s'$ be the label-matching bijections that carry each observable and each preparation of $\mathcal{M}_1$ to the identically labelled one of $\mathcal{M}_2$ (as in Paper III [11], Definition 3.1), and put

$$d_{T,\mathcal{O},S}(\mathcal{M}_1, \mathcal{M}_2) = \sup_{s\in S, t\in[0,T], f\in\mathcal{O}} |f(\Phi^t_{H_1}(s)) - f'(\Phi^t_{H_2}(s'))|.$$

Write $\mathcal{M}_1 \approx_\varepsilon \mathcal{M}_2$ when $d_{T,\mathcal{O},S} \le \varepsilon$.

Theorem 4.3 (Finite-precision equivalence is a tolerance relation)**.** *$\approx_\varepsilon$ is reflexive and symmetric but, for every $\varepsilon > 0$ on ordinary continuous model families, need not be transitive. Taking its transitive closure can identify the entire family.*

Proof. The displayed supremum is a pseudometric. On $H_g = p^2/2 + gq$, prepared at $(0,0)$ and measured by q for duration T, one has $d(\mathcal{M}_g, \mathcal{M}_{g'}) = |g - g'|T^2/2$. Choose three equally spaced g_i with adjacent distance ε and endpoint distance 2ε. Subdivision joins any two g values by finitely many ε-steps, so the transitive closure is total. □

Proposition 4.4 (Finite equivalence horizon)**.** *If X is L-Lipschitz, $\|X - Y\| \le \delta$, and paired trajectories remain in the certified domain, then*

$$|x(t) - y(t)| \le \frac{\delta}{L}(e^{Lt} - 1), \qquad T_\varepsilon = \frac{1}{L}\log(1 + L\varepsilon/\delta).$$

The estimate is sharp for a Hamiltonian saddle.

Proof. The first inequality is Gronwall's estimate [6]. For $H = Lqp$ and $K = Lqp+\delta p$, trajectories from the origin differ in q by exactly $\delta(e^{Lt} - 1)/L$. □

Theorem 4.5 (Exact finite data underdetermine the Hamiltonian)**.** *Let finitely many trajectories of H_0 be observed on a finite horizon, even continuously, at infinite precision and through every smooth observable. The consistent Hamiltonians contain an infinite-dimensional affine family.*

Proof. The finite union Γ of trajectory segments is compact and has empty interior. Choose a chart ball B disjoint from Γ. Every $\chi \in C_c^\infty(B)$ has $d\chi = 0$ on Γ, so $X_{H_0+\chi} = X_{H_0}$ there. Uniqueness makes the observed trajectories and all their observable values identical. Since $C_c^\infty(B)$ is infinite-dimensional, so is the consistent fibre. □

Counterexample 4.6 (Equivalence is not structural identity)**.** An oscillator on $\mathbb{R}^2$ and the same visible oscillator with an unprepared, unmeasured second mode on $\mathbb{R}^4$ produce identical records for all times and precisions, yet their state spaces have different dimensions and are not diffeomorphic. Conversely, Proposition 2.3 shows that a symplectic isomorphism need not preserve fixed laboratory pointers.

New result claimed by this work 4.7 (Observational tolerance is not a congruence)**.** Part-level observational equivalence need not survive interconnection: coupling can enlarge the reachable set into a region on which two previously indistinguishable Hamiltonians differ.

Proof. In Theorem 4.5, place $\operatorname{supp}\chi$ outside the isolated observed orbit. Paper III [11] supplies the explicit witness: the observed part is the unit-frequency oscillator prepared at $(1,0)$ with orbit the unit circle, χ is supported in the ball of radius 0.3 about $(2,0)$, and the driver is a second unit-frequency oscillator prepared at $(4,0)$ coupled by $H_{\text{int}} = \kappa q_A q_B$ with $\kappa = 0.01$. Resonant bilinear coupling transfers amplitude over a beat of period π/κ, carrying the composite trajectory into $\operatorname{supp}\chi$. The two isolated parts remain exactly equivalent, while the composite vector fields differ by $X_\chi \neq 0$, so the composite discrepancy is bounded below by a positive constant. We do *not* claim it can be driven to infinity uniformly by scaling χ: scaling changes the dynamics inside the support and hence the dwell time, and that uniform estimate is proved neither here nor in Paper III. The novelty of this precise reachability formulation is claimed with moderate confidence; a direct antecedent would require downgrading it to an established result. □

5 Failure modes and falsifiable content

Definition 5.1 (Four failure modes)**.** For a model, instrument and record: **F1** is precision failure, a prediction missing the record within its certified domain; **F2** is domain failure, where a trajectory or pointer leaves the certified range; **F3** is idealization failure, where a claimed limiting family does not converge uniformly on the relevant compact set and horizon; **F4** is reconstruction degeneracy, where multiple models fit. F4 includes off-support freedom, level-set reparametrization and hidden structural degrees of freedom.

Counterexample 5.2 (Hamiltonicity depends on interpretation)**.** For $m\ddot{q} = -kq - \gamma m\dot{q}$, identifying $p = m\dot{q}$ gives divergence $-\gamma$, so no autonomous canonical Hamiltonian generates the measured (q, p) flow. With $P = e^{\gamma t} m\dot{q}$, however, $H = e^{-\gamma t} P^2/(2m) + e^{\gamma t} kq^2/2$ generates the same second-order equation. The time-dependent change scales the symplectic form and is not canonical. Thus "is Hamiltonian" is not a property of an un-interpreted differential equation.

Empirical statement 5.3. At $\gamma = 0.1\,\mathrm{s}^{-1}$, the physical (q, p) phase-volume factor after 100 seconds is $e^{-10} \approx 4.5\times10^{-5}$, whereas an autonomous Hamiltonian flow preserves that volume. Given calibrated phase-space reconstruction, this contrast is experimentally decidable.

Interpretation 5.4 (Status of the founding thesis)**.** "Physics is the study of empirically realized compositional mathematical structures" is a definition-like characterization, not a falsifiable empirical proposition: interpreted broadly, any successful physical model can be called realized and some operation can be called composition. The falsifiable residue is narrower: a specified class of systems and processes forms a stated symmetric monoidal structure; empirical realization preserves that structure; and the resulting model predicts records under a declared auxiliary bundle. Theorem 3.2, Established result 3.5, and New result 4.7 are counterexample conditions for those clauses.

Conjecture 5.5 (Open reconstruction claim)**.** For generic finite records and positive tolerance, the consistent set has nonempty interior in an appropriate C^k topology. Theorem 4.5 does not prove this: it treats finitely many finite-horizon trajectories and constructs off-support perturbations, but supplies neither genericity nor a meagreness statement.

6 Synthesis and research contract

Proposition 6.1 (Minimal empirical contract)**.** *A scientifically usable compositional claim in this scope must state:*

1. *the object and process classes, including regularity and completeness domains;*
2. *the proposed product or interconnection and all extra interaction data;*

3. the preparation, observable, unit, pointer and error correspondences;

4. the sense of equivalence and its dependence on $(\varepsilon, T, \mathcal{O}, S)$;

5. closure, clean-intersection and reduction hypotheses;

6. failure criteria separating mismatch, domain exit, singular limit and nonidentifiability.

Without these entries, "empirically realized compositional structure" has no determinate test.

Proof. Items 1 and 5 block the failures in Assumption 1.1, Counterexample 3.3, and Established result 3.5. Item 2 is required by New result 3.4. Items 3 and 4 are required by Proposition 2.3 and Theorem 4.3. Item 6 distinguishes the logically different remedies encoded in Definition 5.1. Omitting any item leaves at least one exhibited counterexample unclassified. □

Corollary 6.2 (Corrected thesis)**.** *Within the pilot scope, the defensible conclusion is: finite-dimensional classical physics uses mathematical structures together with empirically supplied compositional and realization data; neither datum is determined by the Hamiltonian triple, and both remain subject to explicit failure tests.*

Remark 6.3 (Limitations retained)**.** We have not produced a novel empirical prediction, a total category of canonical relations, a universal interconnection operation, or a uniqueness theorem for reconstruction. Long-range forces, dissipation, singular constraints and incomplete flows defeat common idealizations. The two claims labelled new are mathematically supported here and in the topic papers, but literature priority for their exact formulations remains less certain than their correctness.

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